Non-Monotone Comparative Statics in Games of Incomplete Information*

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Abstract

This paper analyses comparative statics for two classes of n-player games of incomplete information with continuous action spaces. The two classes are defined by differences in the payoff and behaviour of the weakest type: the lowest value bidder or highest cost firm. We show that in "weakly competitive games", including all-pay auctions and some oligopoly models, weak types will respond to a stochastically higher distribution of types by playing less aggressively. In "strongly competitive" games, all types play more aggressively. Furthermore, we show that a decrease in dispersion of types, in the sense of a refinement of second order stochastic dominance, although also associated with an increase in competitiveness, may in addition result in less aggressive play by strong types in both strongly and weakly competitive games.

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1 Introduction

Orderings of distributions have long been of interest to economists. Those working in welfare economics rank income or wealth distributions in terms of inequality or dispersion, and tend to use (generalized) Lorenz order, while those working on decision making under risk and uncertainty tend to use stochastic dominance relationships. Stochastic dominance relationships have been also of use in games of incomplete information. However, even a strong ordering of two random variables - first order stochastic dominance - can be insufficient to ensure unambiguous comparisons in some games of incomplete information (see, for example, Maskin and Riley, 2000a, footnote 14). As a consequence, several strengthenings of first order stochastic dominance have been introduced, including the monotone likelihood ratio order used for a wide class of examples (Athey, 2002) and the monotone probability ratio order (also known as conditional stochastic dominance or the reverse hazard rate order) used in auctions (Lebrun, 1998; Maskin and Riley, 2000a). These orderings of distributions allow comparative statics in games of incomplete information involving changes in distributions of general rather than specific functional form.

In this paper, we extend these types of comparative static result in two ways. First, we identify two classes of games that have qualitatively different comparative statics predictions. Specifically, there is a difference in response by weak types, for example, low value bidders or high cost firms, to a change in the distribution of types in the sense of a strong refinement of stochastic dominance. In the equilibrium of "strongly competitive" games (such as standard first-price auctions) even weak types are motivated, so that a stochastically higher distribution of types leads to more aggressive play by all. On the contrary, in "weakly competitive" games (such as all-pay auctions and some oligopoly models) weak types are discouraged, so that in the more competitive environment they compete less hard. For example, in a model of price dispersion due to Bagwell and Wolinsky (2002), a stochastically lower distribution of costs will lead high-cost firms to charge higher prices but low cost firms to decrease prices.

Second, while being powerful analytical tools, monotone orderings are very restrictive, ruling out many interesting cases. Being refinements on first order stochastic dominance, they offer no predictions for changes in the distributions that satisfy second order but not first order dominance. Informally speaking, this involves transformations leading to valuations (or signals, etc.) being "less dispersed" but not necessarily "higher" than before. There has been little work on the comparative statics arising from a change in distributions in terms of dispersion. For example, in oligopoly games incomplete information typically represents a degree of uncertainty about opponents' costs. What happens if there is a decrease in the level of uncertainty? With this question in mind, we employ a refinement of second order stochastic dominance based on the unimodality of the likelihood ratio, introduced by Ramos, Ollero and Sordo (2000). Intuitively, one would expect that such decrease in dispersion of types would lead to uniformly more aggressive play - for example, all firms would price lower than previously. We show that in fact in a an oligopoly model, while most types lower prices, such a change may result in low cost firms raising prices. Similarly, in first price auctions, a reduction in dispersion in the sense of the new orderings prompts most types to bid more aggressively, but the highest types may bid less.

The hypothesis that more precise information should lead to uniformly more aggressive bidding and higher selling prices has been investigated in the context of common values by Kagel and Levin (1986) and in subsequent literature for specific functional forms of preferences and distributions of signals. More recently, Goeree and Offerman (2003) investigated the effects of more precise information on the competitive bidding in a framework that nests both private and common value cases. Yet, the major drawback of this literature is that providing agents with "more precise information" has been frequently analyzed by considering two uniform distributions with different support. While being analytically convenient, this assumption is restrictive. We show that the unimodal ratio orderings could serve as an alternative technique allowing to analyze more general pairs of distributions.

It is worth reminding that measures of stochastic dominance are not confined to the economics of information. Since the famous work of Atkinson (1970) they have also been important in the literature on social welfare and the comparisons of income distributions (see Lambert (1989) for a survey). However, the ordering more commonly used in this literature is (generalized) Lorenz dominance, even though it is equivalent to second order stochastic dominance (Thistle, 1989), and, thus, both measures can be interpreted in terms of inequality. More recently, income inequality and games of incomplete information have been considered together (Hopkins and Kornienko, 2004; Samuelson, 2004) in the context of strategic social interaction, where the question has been whether increasing equality leads to greater social competition. It is hoped that this paper will be of some interest to researchers in both fields as well as in their intersection.

We start with an introduction of our classification of games of incomplete information into strongly and weakly competitive games. The next two sections introduce some technical tools necessary for our comparative statics results. Section 3 presents a useful theorem which is, in effect, a two-equation counterpart of Karlin's (1968) variation-diminishing property, while Section 4 briefly surveys refinements of second-order stochastic dominance orderings. In Section 5, we use the new refinements to analyze the effect of changes in dispersion in games of incomplete information. Section 6 gives several examples.

2 Strongly and Weakly Competitive Games of Incomplete Information

In this section, we introduce two classes of games, which we call strongly competitive and weakly competitive. They are differentiated on the basis of a seemingly innocuous condition on the payoff of the lowest ranked agent. However, as we will see, this small difference will lead to quite different outcomes, both in terms of equilibrium behaviour and in comparative statics.

We consider symmetric games of incomplete information with n players. Each player has a type z drawn from a common distribution $F(z)$, which is twice differentiable with strictly positive density on its support $[\underline{z}, \bar{z}]$. Each agent takes an action x, which in different games could be a bid, a price or a choice of effort. This is chosen from a continuous action space which we take to be some subset of the real line. Strategies will therefore be of the form $x(z)$, a mapping from type to action. We go on to consider the effects of changes in the distribution $F(z)$ on the symmetric equilibrium strategy.

For simplicity, we examine the class of games which possess a single "prize" for the highest bid or action. This is sufficiently broad to include a wide range of auctions and contests. If an agent of type z wins with bid x, she gains a payoff $U(z-x)$. Otherwise, she has a payoff of $U(-Ix)$, where I an indicator function that is one when a losing agent pays his bid (as in all-pay auctions and contests) and zero if he does not (as in standard auctions). We assume that $U(\cdot)$ is twice continuously differentiable with $U' > 0$ and $U'' \leq 0$. Suppose all agents adopt the same strictly increasing differentiable strategy $x(z)$, then the expected utility of an agent of type z who bids $x(\hat{z})$, that is, as if she had type $\hat{z}$ will be

$$V(x(\hat{z}), z, z_{-i}) = F^{n-1}(\hat{z})U(z - x(\hat{z})) + \left(1 - F^{n-1}(\hat{z})\right) U(-Ix(\hat{z})) \tag{1}$$

Differentiating with respect to $\hat{z}$, setting $\hat{z}$ to z and rearranging, we obtain the following differential equation

$$x'(z) = \frac{h(z)(U(z-x) - U(-Ix))}{U'(z-x)F^{n-1}(z) + U'(-Ix)(1 - F^{n-1}(z))}, \tag{2}$$

where $h(z) = (n-1)f(z)F^{n-2}(z)$. Solutions to this differential equation will constitute symmetric equilibria for games of this class.

The boundary condition for this differential equation will be the equilibrium strategy of the weakest agent, the one with type $\underline{z}$. In fact, as we will now see, it is possible to divide these games of incomplete information into two broad classes on the basis of her behaviour. In the first class, which includes the classic first price auction, in equilibrium weak types bid (close to) the maximum possible. In the second class, in complete contrast, in equilibrium the lowest type bids nothing or supplies zero effort.

The criterion for determining whether a game is weakly or strongly competitive is the nature of the payoff to the weakest type. In a symmetric equilibrium, the lowest type always comes last. We consider the payoffs for that lowest type $\underline{z}$. From (1), we have in symmetric equilibrium $V(x(\underline{z}), \underline{z}, z_{-i}) = U(-Ix(\underline{z}))$. If this payoff is constant and fixed, that is, if $I = 0$, then we call the game "strongly competitive". For example, in a standard first price auction, the lowest bidder will receive a zero payoff. If the equilibrium payoff of the lowest type depends on her action, then we call the game "weakly competitive". For example, in an all-pay auction, the lowest bidder will never win the prize, but her payoff will be $U(-x)$, where x is her bid. We now show that the equilibrium behaviour of the low types in the two classes of game is quite different. Let $\underline{x}$ be the minimum feasible bid, typically zero.

Lemma 1 *Consider an $n-$player bidding game of incomplete information with types drawn from a continuous non-zero density on $[\underline{z}, \bar{z}]$. Suppose $I = 0$, then, for any symmetric strictly increasing equilibrium strategy $x(z)$ necessarily $\lim_{z \to \underline{z}} x(z) = \underline{z}$. The weakest type is consequently indifferent between any action in the range $[0, \underline{z}]$. Suppose $I = 1$, then in a symmetric equilibrium $x(\underline{z}) = \underline{x}$.*

Proof: If $I = 0$, suppose that $\lim_{z \to \underline{z}} x(z) = x_0 < \underline{z}$. Then, the lowest type could choose an action $\hat{x}$ in $(x_0, \underline{z})$ such that $F(x^{-1}(\hat{x})) > 0$. This represents a profitable deviation. If indeed $\lim_{z \to \underline{z}} x(z) = \underline{z}$, then $V(x, \underline{z}, z_{-i}) = 0$ for any feasible action $x \in [0, \underline{z}]$, and so the weakest type is indifferent. If $I = 1$, the equilibrium payoff to the lowest type, that is $U(-x(\underline{z}))$ is decreasing in x. So, clearly $\underline{x}$ is optimal. ∎

While the above result indicates that it is possible for the weakest agent to choose a bid of zero in both strongly and weakly competitive games, in general play by weak agents will be very different. By continuity of the equilibrium strategy, in weakly competitive games, agents with types close to $\underline{z}$ will bid close to zero, but in strongly competitive games, they will bid close to the maximum rational amount. Furthermore, as we will see later in Section 5, the comparative static effect of an increase in competitiveness is opposite in the two classes. In weakly competitive games, low types supply even less effort in a more competitive environment.

Another consequence of the above Lemma is that in the strongly competitive case, in a sense equilibrium cannot be unique. As is well known in the analysis of first price auctions (see, for example, Maskin and Riley (2003)), behaviour at the lower boundary is not uniquely defined, as the weakest agent always makes zero profit and is indifferent over all possible bids. However, behaviour on the interior of the type space can be shown to be unique.

Proposition 1 *The differential equation (2) with boundary condition $\lim_{z \to \underline{z}} x(z) = \underline{z}$ for $I = 0$ and boundary condition $x(\underline{z}) = 0$ for $I = 1$ has a unique solution on $(\underline{z}, \bar{z})$ and this is the unique symmetric equilibrium on $(\underline{z}, \bar{z})$.*

Proof: In the Appendix. ∎

There are similar considerations for procurement auctions and oligopoly games. Suppose n firms each have constant marginal cost c but the exact level of that cost is private information. Each is an independent draw from a distribution $F(c)$ with a continuous positive density on $[\underline{c}, \bar{c}]$. The firms compete on price in a simultaneous move game. We assume there is a finite maximum price $\bar{p}$.[1] Let $D_1(p)$ be the demand to the firm who names the lowest price and $D_0(p)$ be the demand otherwise. We assume that D_1 is discontinuously higher than D_0, or $\lim_{p \uparrow \hat{p}} D_1(p) > D_0(\hat{p}) \geq 0$ for all $\hat{p} \leq \bar{p}$. The classic example of such a market is Bertrand competition, where D_1 is strictly positive but D_0 is zero. We give another oligopoly model later in Section 6.2, where both D_0 and D_1 are positive. In either case, the expected profit for a firm charging $p(\hat{c})$ when all other use the strictly increasing continuous strategy $p(c)$ will be

$$V(p(\hat{c}), c, c_{-i}) = (1 - F(\hat{c}))^{n-1} (p(\hat{c}) - c) D_1(p(\hat{c})) + \left(1 - (1 - F(\hat{c}))^{n-1}\right) (p(\hat{c}) - c) D_0(p(\hat{c})). \tag{3}$$

We find again that in strongly competitive games, the weak agents, here the high cost firms, are in a desperate situation and are reduced to charging at cost. In contrast, in weakly competitive games, weak agents charge the maximum price.

Lemma 2 *Consider an $n-$player pricing game of incomplete information with types drawn from a continuous non-zero density on $[\underline{c}, \bar{c}]$. Suppose $D_0(p)$ is zero for all p. Then, for any symmetric strictly increasing equilibrium strategy $p(c)$ necessarily $\lim_{c \to \bar{c}} p(c) = \bar{c}$. Suppose $D_0(p)$ is not zero, then in a symmetric equilibrium $p(\bar{c}) = \bar{p}$.*

Proof: This is readily derivable from the proof to Lemma 1. ∎

3 A Useful Theorem

In this section, we present a quite general result that will be useful in the comparative static analysis of games of incomplete information. Typically, in auctions, signalling games and pricing games, where players choose an action from a continuum, the equilibrium will be a solution to a differential equation. It is not always possible to find analytic solutions to such equations. However, it can be possible to obtain results on comparative statics, if one can place some restrictions on the behaviour of the solutions to the relevant differential equation. Suppose there are two solutions, each representing a different value of an exogenous parameter or a different distribution of types. The theorems in this section enable the comparison of the two solutions by limiting the number of times that they can cross.

We have already seen that there exist games of incomplete information where the equilibrium strategy $x(z)$ can be calculated as a solution to a differential equation (2). If we look at two specific cases, either the standard auction case of $I = 0$ or when $I = 1$ but agents are risk neutral, then we can write the corresponding differential equation in the following form:

$$x'(z) = \psi(x(z), z) m(z), \quad x(\underline{z}) = \underline{x}. \tag{4}$$

[1] For example, consumers have a unit demand for the good, provided the price does exceed their reservation price $\bar{p}$.

We assume ψ and m are continuously differentiable positive functions. That is, $\psi(\cdot,\cdot) > 0, m(\cdot) > 0$. This implies that $x'(z) > 0$ on $(\underline{z}, \bar{z})$, so that the solution $x(z)$ is increasing. We also assume $\psi_1 \leq 0$, ψ is decreasing in x. This is natural in many games, where $\psi(\cdot)$ will be derived from an agent's utility function, and x represents a bid in an auction, or effort in a contest. The function $m(z)$ will typically depend on exogenous parameters and/or the distribution of types. Changes in either will, therefore, change $m(z)$. The theorem below shows the implications of such changes for the equilibrium strategy $x(z)$.[2] Finally, let $S(f)$ be the number of sign changes of the function $f(z)$.

Theorem 1 *Let $x_F(z)$ and $x_G(z)$ denote the solutions to the differential equation (4) for functions $m_F(z)$ and $m_G(z)$, respectively, with $x_F(\underline{z}) = x_G(\underline{z})$. Then if $m_F(z) - m_G(z) > 0$ on some interval $(\underline{z}, \underline{z}+\delta), \delta > 0$, then $x_F(z) - x_G(z) > 0$ on this interval as well. Moreover, if $x_F(z)$ and $x_G(z)$ do cross for $z \geq \underline{z}+\delta$, then at every point of crossing $x_F(z)$ crosses $x_G(z)$ from above (below) only if $m_F(z) - m_G(z) < (>)0$ at the point of crossing, so that $S(x_F(z) - x_G(z)) \leq S(m_F(z) - m_G(z))$.*

Proof: In the Appendix. ∎

We also present a complementary result for a different class of games. Typically, they involve firms whose strategies are mappings from costs to prices. This involves several differences from the previous setup. First, we adopt appropriate notation, so that a player's type is now c and her action p. Second, we assume that ψ is increasing (and not decreasing) in the player's action p. Finally, as we have seen in the previous section, the boundary condition will typically be set at the upper boundary, rather than the lower. So, let $p(c)$ be the solution over some interval $[\underline{c}, \bar{c}]$ to the following differential equation:

$$p'(c) = \psi(p(c), c)m(c), \quad p(\bar{c}) = \bar{p} \tag{5}$$

where ψ and m are continuously differentiable positive functions. That is, $\psi(\cdot,\cdot) > 0, m(\cdot) > 0$. We also assume $\psi_1 \geq 0$, ψ is increasing in p. The proof for the following theorem can readily be derived from the proof to the preceding result.

Theorem 2 *Let $p_F(c)$ and $p_G(c)$ denote the solutions to the differential equation (5) for functions $m_F(c)$ and $m_G(c)$, respectively, with $p_F(\bar{z}) = p_G(\bar{z})$. Then if $m_F(c) - m_G(c) > 0$ on some interval $(\bar{z}-\delta, \bar{z}), \delta > 0$, then $p_F(c) - p_G(c) < 0$ on this interval. Moreover, if $p_F(c)$ and $p_G(c)$ do cross for $c \leq \bar{c}-\delta$, then at every point of crossing $p_F(z)$ crosses $p_G(z)$ from below (above) only if $m_F(z) - m_G(z) > (<)0$ at the point of crossing, so that $S(p_F(z) - p_G(z)) \leq S(m_F(z) - m_G(z))$.*

[2] This theorem is, in effect, a two-equation counterpart of Karlin's (1968) variation-diminishing property.

4 Ordering Distributions in Terms of Dispersion

Ordering distributions in terms of stochastic dominance is now a common tool in the economics of information. However, the concentration up to now has been on first order stochastic dominance and its refinements. Clearly, those working on income distributions, since the seminal work by Atkinson (1970), have had greater interest in second order stochastic dominance (equivalent to the generalized Lorenz order - see Thistle (1989)), which allows ordering of distributions in terms of dispersion or inequality. In this section, we briefly survey the results for a refinement of second order stochastic dominance, introduced in the context of the analysis of income distributions, which we will then go on to use in comparative statics.

In what follows, we consider two distinct non-negative variables X and Y with finite means μ_X and μ_Y respectively, having distribution functions F and G, respectively, with F and G both having support $[\underline{z}, \bar{z}]$ with $0 \leq \underline{z} < \bar{z}$. Assume that F and G are twice continuously differentiable and the densities f and g are strictly positive on the corresponding supports. We employ the following definition of unimodality.[3]

Definition 1 *A function $f(z)$ is unimodal around $\hat{z}$ if $f(z)$ is strictly increasing for $z < \hat{z}$ and $f(z)$ is strictly decreasing for $z > \hat{z}$.*

The following order of distributions was first introduced by Ramos, Ollero and Sordo (2000). They show that this order implies second order stochastic dominance (equivalently generalised Lorenz dominance).

Definition 2 *Two distributions F, G satisfy the Unimodal Likelihood Ratio (ULR) order and we write $F \succ_{ULR} G$ if the likelihood ratio $L(z) = f(z)/g(z)$ is unimodal and $E[X] \geq E[Y]$.* [4]

In simpler terms, if $F \succ_{ULR} G$ then distribution F is either stochastically higher than G or it is more equal. Consider a simple example. Suppose $G(z)$ is a uniform distribution so that its density $g(z)$ is a constant, then $L(z)$ will be unimodal if $f(z)$ is unimodal, that is, it is less dispersed than $g(z)$. It is well-known (see, for example, Dharmadhikari and Joag-Dev (1988)) that all logconcave functions are unimodal.[5] Thus, if $\log L(z)$ is concave and $\mu_X \geq \mu_Y$, then $F \succ_{ULR} G$. From our definition of unimodality, there is a unique value of z which we denote $\hat{z}_L$ which maximizes the likelihood ratio $L(z)$, with

[3] This is a slight strengthening of standard definitions of unimodality - for example, by Dharmadhikari and Joag-Dev (1988, Chapter 1) and by An (1998). In the first source, a function $f(z)$ is unimodal if $\int_{\underline{z}}^{z} f(t)dt$ is convex on $(\underline{z}, \hat{z})$ and concave on $(\hat{z}, \bar{z})$. In the second, the function $f(z)$ has to satisfy the following: for all $\delta > 0$, the set $D_\delta = \{z \in \Omega : f(z) \geq \delta\}$ is a convex set in Ω.

[4] Note that the condition on the means rules out the possibility that the mode is at the lower bound and that Y first order dominates X.

[5] For review of logconcave and logconvex functions see An (1998).

$\hat{z}_L \leq \bar{z}$. If the mode of the ratio is located at the upper bound, that is, $\hat{z}_L = \bar{z}$, we arrive at a monotone order as a special case.

Definition 3 *The two distributions F, G satisfy the Monotone Likelihood Ratio (MLR) order and we write $F \succ_{MLR} G$, if the ratio of their densities $L(z)$ is strictly increasing.*

Milgrom (1981) introduced the MLR order to the economics of information. More recently, Athey (2002) employs the MLR order to obtain monotone comparative statics in games of incomplete information. As Milgrom (1981) points out, many well known families of distributions, for example, the normal and the exponential satisfy the MLR order. A similar set of families of distributions satisfy ULR order. One can easily verify that, for example, if F and G are both normal or both lognormal, with $\mu_X \geq \mu_Y$ and with F having strictly lower standard deviation then $F \succ_{ULR} G$.

It is well-known that the MLR order implies first order stochastic dominance and other refinements of first order stochastic dominance, such as the hazard rate order and the reverse hazard rate order (see, for example, Krishna (2002, Appendix B). Similar relationships can be shown for the ULR order. Define the following ratios[6]

$$P(z) = \frac{F(z)}{G(z)}, \; H(z) = \frac{1 - F(z)}{1 - G(z)}. \tag{6}$$

Proposition 2 *If $L(z)$ is unimodal with maximum at $\hat{z}_L$ then $P(z)$ is unimodal with a maximum at $\hat{z}_P \geq \hat{z}_L$ and $H(z)$ is unimodal with a maximum at $\hat{z}_H \leq \hat{z}_L$.*

Proof: This is Metzger and Rüschendorf (1991, Theorem 2.3 and 2.3 (c)). ∎

The ratio $\sigma(z) = f(z)/F(z)$ is known as the "reverse hazard rate" in the statistics literature (see, for example, Shaked and Shanthikumar (1994)). Note that if $P'(z) > 0$ then

$$\sigma_F(z) = \frac{f(z)}{F(z)} > \frac{g(z)}{G(z)} = \sigma_G(z). \tag{7}$$

There is a similar relation between $H(z)$ and the hazard ratio, which for a distribution function $F(z)$ is, of course, defined as $\lambda(z) = f(z)/(1 - F(z))$. That is, it is the ratio of the density to the survival function, $1 - F(z)$. Note that if $H'(z) > 0$ then

$$\lambda_F(z) = \frac{f(z)}{1 - F(z)} < \frac{g(z)}{1 - G(z)} = \lambda_G(z). \tag{8}$$

Therefore, combined with Proposition 2, these relations lead to the following corollary, which will prove useful for comparative statics.

Corollary 1 *Suppose $F \succ_{ULR} G$ then (i) $\sigma_F(z) > \sigma_G(z)$ almost everywhere on $(\underline{z}, \hat{z}_P)$, and $\sigma_F(z) < \sigma_G(z)$ almost everywhere on $(\hat{z}_P, \bar{z})$; (ii) $\lambda_F(z) < \lambda_G(z)$ almost everywhere on $(\underline{z}, \hat{z}_H)$, and $\lambda_F(z) > \lambda_G(z)$ almost everywhere on $(\hat{z}_H, \bar{z})$.*

[6] The monotonicity of ratio $P(z)$ was considered by Maskin and Riley (2000a) under the name of conditional stochastic dominance.

5 Comparative Statics

In this section we will show how ratio orderings of distributions allow comparative static predictions in many games of incomplete information. Some work has been done on comparative statics for auctions by Lebrun (1998) and Maskin and Riley (2000a), and, for a wider class of examples, by Athey (2002). For a very general specifications for the primitives, that is, preferences and the distributions of types, these researchers derive sufficient conditions for the existence of monotone comparative statics - conditional stochastic dominance and monotonicity of the likelihood ratio, respectively. In other words, a "higher" distribution of valuations in a sense of either of the orderings should lead to a uniformly more aggressive bidding.

Here, we extend this type of result in two ways. First, we show that in many games of incomplete information, such a monotone shift in types is not in fact sufficient for monotone comparative statics. For example, there are plausible models of oligopoly where a stochastically lower distribution of costs will lead some firms to charge higher prices. This is not to say, however, that there are no meaningful results. Rather, we give simple conditions for when games permit monotone comparative statics and, in the games that do not, we identify which classes of agents will adopt higher strategies and which lower.

Second, we allow for a different type of change in the distribution of types which potentially has a number of applications. What happens if the distribution of types becomes less dispersed? For a private value auction, this would mean that the group of bidders becomes more homogenous. The obvious hypothesis is that bidding will be more competitive. This is certainly the case for equilibrium bidding functions calculated for particular functional forms in auctions with affiliated values in Kagel and Levin (1986). However, we show that in general this is not true, even under quite strong regularity conditions. That is, there are plausible circumstances in which more precise information will induce some agents to bid less.

Earlier we introduced two classes of games, strongly and weakly competitive. We will now show that, in addition to the different boundary conditions, these classes of games have qualitatively different comparative statics. Specifically, we examine the effect of a more competitive distribution of types in the sense of the ULR order, introduced in the previous section, on equilibrium strategies. Remember that dominance the ULR order implies that the dominant distribution is either higher or less dispersed than the dominated. We find that such a change in the distribution will lead to higher effort for low types in strongly competitive games (where $I = 0$), but to lower effort for low types in games that are weakly competitive (where $I = 1$).

Theorem 3 *Suppose there are two distributions F, G such that $F(z) \succ_{ULR} G(z)$. Let $x_F(z)$ and $x_G(z)$ be the corresponding solutions to the differential equation (2). Then (i) if $I = 0$ (so that the game is strongly competitive), $x_F(z) > x_G(z)$ on $(\underline{z}, \hat{z}_P)$ where $\hat{z}_P$ is where the ratio $P(z) = F(z)/G(z)$ has its maximum; (ii) if $I = 1$ (so that the* *game is weakly competitive), there exists an* $\epsilon > 0$ *such that* $x_F(z) < x_G(z)$ *on* $(\underline{z}, \underline{z}+\epsilon)$.

Proof: In the Appendix. ∎

Since the monotone likelihood ratio (MLR) order is a special case of the ULR order, the above result implies that in weakly competitive games even a strongly monotone increase in types will induce a reduction in bidding or effort by some agents. However, note that in strongly competitive games the above result implies a strong competitive response. For example, under the MLR order, the point $\hat{z}_P$ is at the upper bound $\bar{z}$. That is, in strongly competitive games, $x_F(z) > x_G(z)$ almost everywhere. It is, therefore, even more surprising that this is not the case in all games. The following section gives several examples, and develops what exactly does happen in weakly competitive games.

6 Some Examples

In this section we consider a few examples of different classes of games and demonstrate that they indeed have different comparative statics.

6.1 Class 1: Strongly Competitive Games

In this subsection, we examine some familiar examples of strongly competitive games of incomplete information. Here, the lowest type makes zero profit, and a monotone increase in the distribution of types leads to a monotone increase in the equilibrium strategy.

The first example is the independent private value first price auction where the value a bidder places on the object for sale is simply her type, z. Assume the type for each agent is an independent draw from the same distribution $F(z)$ with support $[\underline{z}, \bar{z}]$. In this model, if all other agents adopt the same strictly increasing strategy $x(z)$, then the expected utility of an agent of type z who bids $x(\hat{z})$ would be

$$V(x(\hat{z}), z, z_{-i}) = U(z - x(\hat{z}))F^{n-1}(\hat{z})$$

where $U(\cdot)$ is the agent's Von Neumann-Morgenstern utility function, with $U(0)$ normalised to zero. We differentiate the above expression with respect to $\hat{z}$, set the resulting derivative to zero, and note that a symmetric equilibrium then requires that we set $x(\hat{z}) = x(z)$. The first order conditions then give rise to the differential equation

$$x'(z) = \frac{(n-1)f(z)}{F(z)} \frac{U(z-x)}{U'(z-x)}, \tag{9}$$

which is a special case of the differential equation (4) with $\psi(x(z), z) = U(z-x)/U'(z-x)$ and $m(z) = (n-1)f(z)/F(z)$, (a positive multiple of) the reverse hazard ratio. The standard boundary condition in the independent private value case is that $\lim_{z \to \underline{z}} x(\underline{z}) = \underline{z}$. Note that if $U(\cdot)$ is strictly increasing and (weakly) concave then $\psi_1 < 0$ and $\psi_2 > 0$.

We show next that the unimodal likelihood ratio order can be used to obtain comparative statics for general utility functions. In particular, it implies that more accurate information will always lead to more aggressive bidding for those with relatively low signals, and may lead to less aggressive bidding for only those with relatively high signals. To be more precise, if $F(z) \succ_{ULR} G(z)$, so that $F(z)$ and $G(z)$ cross at most once at some point $\tilde{z}$ on the interior of their support, then the corresponding bidding functions will cross no more than once and only to the right of $\tilde{z}$. Moreover, if the functions cross, they cross to the right of the maximum of $P(z) = F(z)/G(z)$. In other words, equilibrium bidding functions $x_F(z)$ and $x_G(z)$ behave as shown in Figure 1.

Proposition 3 *Suppose $x_F(z)$ and $x_G(z)$ are the equilibrium bidding functions for distributions $F(z)$ and $G(z)$, respectively. If $F(z) \succ_{ULR} G(z)$, then either $x_F(z) > x_G(z)$ almost everywhere, or there exists a point $z^* > \operatorname{argmax} P(z) = \hat{z}_P$ such that $x_F(z^*) = x_G(z^*)$, $x_F(z) > x_G(z)$ for all $z \in (\underline{z}, z^*)$ and $x_F(z) < x_G(z)$ for all $z \in (z^*, \bar{z})$.*

Proof: The proof follows directly from Theorem 1 by observing that differential equation (16) is a special case of differential equation (4) and that, by Corollary 1, $\sigma_F(z) > \sigma_G(z)$ on $(\underline{z}, \hat{z}_P)$ and $\sigma_F(z) < \sigma_G(z)$ on $(\hat{z}_P, \bar{z})$. ∎

The intuition behind the failure of monotonicity disclosed in Proposition 3 can be expressed in the following tradeoff. As the distribution of types becomes more compressed, the marginal return to raising one's bid rises, inducing more aggressive bidding. However, for those with types above $\tilde{z}$, the point of intersection of $F(z)$ and $G(z)$, the probability of winning has risen as $F(z) > G(z)$ for $z > \tilde{z}$. This has the opposite effect. Thus monotonicity may fail for some z above $\tilde{z}$. This behaviour is illustrated in Figure 1. One can also remark that this second effect is increasing with the competitiveness of the auction, that is, the more risk averse are bidders or the larger the number of participants.

Lebrun (1998) and Maskin and Riley (2001a) showed that if the two distributions satisfy the monotone probability ratio property, one can obtain monotone comparative statics in asymmetric first-price auctions.[7] Similarly, Athey (2002) uses the monotone likelihood ratio order to obtain monotone comparative statics. The corollary below is a similar result for symmetric first-price auctions. If the maximum of the likelihood ratio is at the upper bound, i.e. $\hat{z}_L = \bar{z}$, then the ratio is monotone and the above proposition implies that the solutions will not cross.

Corollary 2 *Suppose $x_F(z)$ and $x_G(z)$ are the equilibrium bidding functions for distributions $F(z)$ and $G(z)$, respectively. If $F(z) \succ_{MLR} G(z)$, then $x_F(z) > x_G(z)$ almost everywhere.*

[7] The monotone probability ratio order, also known as the reverse hazard rate order, is implied by the monotone likelihood ratio order. See Krishna (2002, Appendix B).

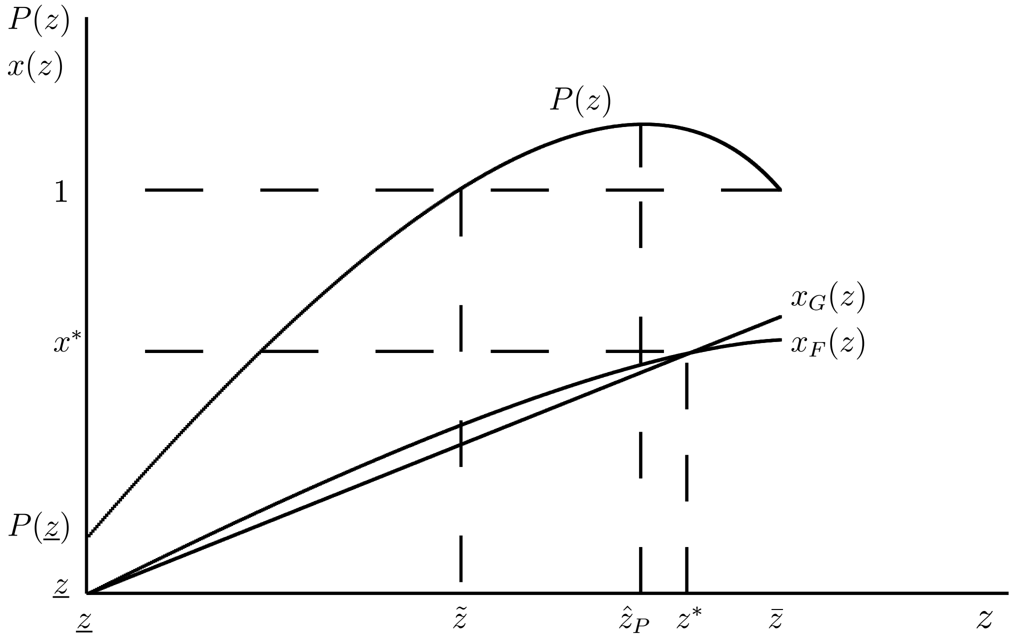

Figure 1: Comparative Statics for a Strictly Competitive Game

Another special case of this approach is the effect of a change in an exogenous parameter. As a very simple example, an increase in n the number of bidders in the first price auction will lead to an upward shift in the function $m(z)$, as in this context $m(z)$ is defined as $(n-1)f(z)/F(z)$. Then, by Theorem 1, the equilibrium bidding function must rise everywhere on $(\underline{z}, \bar{z}]$. This result, that an increase in the number of bidders in the first price auction will lead to more aggressive bidding is well-known. However, it does illustrate that a parameter shift can be treated as a special and simple case of these comparative statics techniques.

Another example is Bertrand competition with private information about costs, a game which is strategically similar to a procurement auction (see Spulber (1995) for a more complete treatment). In both cases, the player who names the lowest price wins the prize (in the auction, the contract; in Bertrand competition, sales). Assume there are n firms that compete on price to sell to N consumers. Each firm has private information about its marginal cost c, which is an independent draw from the distribution $F(c)$ with support $[\underline{c}, \bar{c}]$. Each consumer seeks to buy one unit of the good, if the price does not exceed a common reservation price $\bar{p}$. Expected profits for a firm with cost c from charging a price $p(\hat{c})$ when the other sellers use the symmetric strictly increasing strategy $p(c)$ are thus

$$V(p(\hat{c}), c, c_{-i}) = N(p(\hat{c}) - c)\,(1 - F(\hat{c}))^{n-1}$$

This is Case 1, as in symmetric equilibrium, the highest cost firm cannot win the contest and must make zero profits. A symmetric equilibrium will therefore satisfy the differential equation

$$p'(c) = (p - c)(n - 1)\left(\frac{f(c)}{1 - F(c)}\right), \quad \lim_{c \to \bar{c}} p(c) = \bar{c} \tag{10}$$

So, in this case $\psi(p, c) = (p - c)$ and $m(z) = (n - 1)f(c)/(1 - F(c))$. The comparative statics of this game are similar but reversed to those in the first price auction. Consider a shift in the distribution of costs so that, for example, $F(c) \succ_{MLR} G(c)$. The next result shows that such an increase in costs will lead to higher prices.

Proposition 4 *Suppose $p_F(c)$ and $p_G(c)$ are the equilibrium price functions for distributions $F(c)$ and $G(c)$, respectively. If $F(c) \succ_{MLR} G(c)$, then $p_F(c) > p_G(c)$ almost everywhere.*

Proof: If $F(c) \succ_{MLR} G(c)$ by Corollary 1, $f(c)/(1 - F(c)) < g(c)/(1 - G(c))$ on $(\underline{c}, \bar{c})$. The result then follows directly from Theorem 2. ∎

6.2 Class 2: Weakly Competitive Games

In this section we examine several weakly competitive games of incomplete information, where in equilibrium the lowest or weakest type makes no effort at all. Furthermore, we will show that in this class of game an increase in competitiveness in the sense of a stochastically higher distribution of types will lead to lower effort by low types.

For example, consider the following all-pay auction with private information.[8] A bidder of type z, receives a payoff of $z - x$ if her bid x is the highest, and a payoff of $-x$ otherwise. Thus, an agent of type z bidding $x(\hat{z})$ when all other agents bid according to the strictly increasing strategy $x(z)$ will obtain an expected utility

$$V(x(\hat{z}), z, z_{-i}) = F^{n-1}(\hat{z})z - x(\hat{z})$$

Note that, in a symmetric equilibrium, the agent with the lowest type $\underline{z}$ will gain an expected utility of $-x$. Clearly, the optimal bid in this case is zero. This game, therefore, belongs in Class 2. We differentiate the above expression for expected utility with respect to $\hat{x}$ and set the derivative to zero. Then, in a symmetric equilibrium, $\hat{x} = x(z)$, giving rise to the following differential equation,

$$x'(z) = zh(z), \quad x(\underline{z}) = 0, \tag{11}$$

where, again $h(z) = (n - 1)f(z)F^{n-2}(z)$. So, in this case $\psi(x, z) = z$ and $m(z) = h(z)$.

Alternatively, in a special case of the contests analysed by Moldavanu and Sela (2001), n agents compete for a prize with fixed common value W. Each agent pays a

[8] See Krishna and Morgan (1997) for a more general treatment.

cost cx to produce output x. The prize is awarded to the agent with the highest output. Let $c = 1 - z$, where z is the agent's type which is an independent draw from $F(z)$ with support $[\underline{z}, \bar{z}]$ with $\bar{z} < 1$. Thus, an agent choosing $x(\hat{z})$ when all others adopt the strictly increasing strategy $x(z)$ will obtain an expected utility

$$V(x(\hat{z}), z, z_{-i}) = F^{n-1}(\hat{z})W - (1-z)x(\hat{z}).$$

A symmetric equilibrium in increasing strategies will therefore be a solution to the differential equation

$$x'(z) = \frac{W}{1-z}h(z), \quad x(\underline{z}) = 0 \tag{12}$$

So, in this case $\psi(x, z) = W/(1-z)$ and again $m(z) = h(z)$.

Proposition 5 *Suppose $x_F(z)$ and $x_G(z)$ are the equilibrium bidding functions arising from the differential equations (10) or (11) for distributions $F(z)$ and $G(z)$, respectively. If $F(z) \succ_{ULR} G(z)$, then $h_F(z)$ crosses $h_G(z)$ at least once and possibly twice. This implies that $x_F(z) < x_G(z)$ on $(\underline{z}, \hat{z}_-]$ where $\hat{z}_-$ is the first crossing point of $h_F(z)$ and $h_G(z)$. Furthermore, $x_F(z)$ crosses $x_G(z)$ at least once and from below so that $x_F(\tilde{z}) > x_G(\tilde{z})$ where $\tilde{z}$ is the unique crossing point of $F(z)$ and $G(z)$. If there is a second crossing of $h_F(z)$ and $h_G(z)$, then $x_F(z)$ may cross $x_G(z)$ from above on $(\hat{z}_+, \bar{z})$ where $\hat{z}_+$ is the second crossing point of $h_F(z)$ and $h_G(z)$.*

Proof: Note that in the proof of Theorem 3 we established that $h_F(z)/h_G(z)$ was strictly increasing on $(\underline{z}, \hat{z}_L)$, and that that $h_F(z)$ crosses $h_G(z)$ exactly once and from below. The first claim then follows from Theorem 1.

The solutions $x_F(z)$ and $x_G(z)$ must cross. Note that from (10) and (11), we have $x(\tilde{z}) = \int_{\underline{z}}^{\tilde{z}} \phi(z) dF^{n-1}(z)$ with $\phi(\cdot)$ an increasing function (z and $1/(1-z)$ respectively). By integration by parts $x(\tilde{z}) = \phi(\tilde{z})F(\tilde{z}) - \int_{\underline{z}}^{\tilde{z}} F^{n-1}(z)\phi'(z)dz$. Since $F^{n-1}(z) < G^{n-1}(z)$ on $(\underline{z}, \tilde{z})$ with equality at $\tilde{z}$ the result follows. ∎

This comparative static result is illustrated in Figure 2. This shows an example where $F(z)$ represents a more compressed distribution than $G(z)$. The above result then suggest that low value bidders will bid less under the higher or more compressed distribution $F(z)$, but that the bidding function must cross over before the two distribution functions do so at $\tilde{z}$. A further crossing is illustrated in the Figure. This is possible as $h_F(z) < h_G(z)$ for $z > \hat{z}_+$, but is not guaranteed.

Another example is given by Bagwell and Wolinsky (2002) who consider an incomplete information version of the Varian (1980) model of price dispersion. There are n firms that compete on price to sell to N consumers. Each consumer seeks to buy one unit of the good, if the price does not exceed a common reservation price $\bar{p}$. A proportion q of consumers are uninformed and purchase from a randomly chosen seller. The other $1 - q$ only buy from the lowest priced firm. In the version of Bagwell and

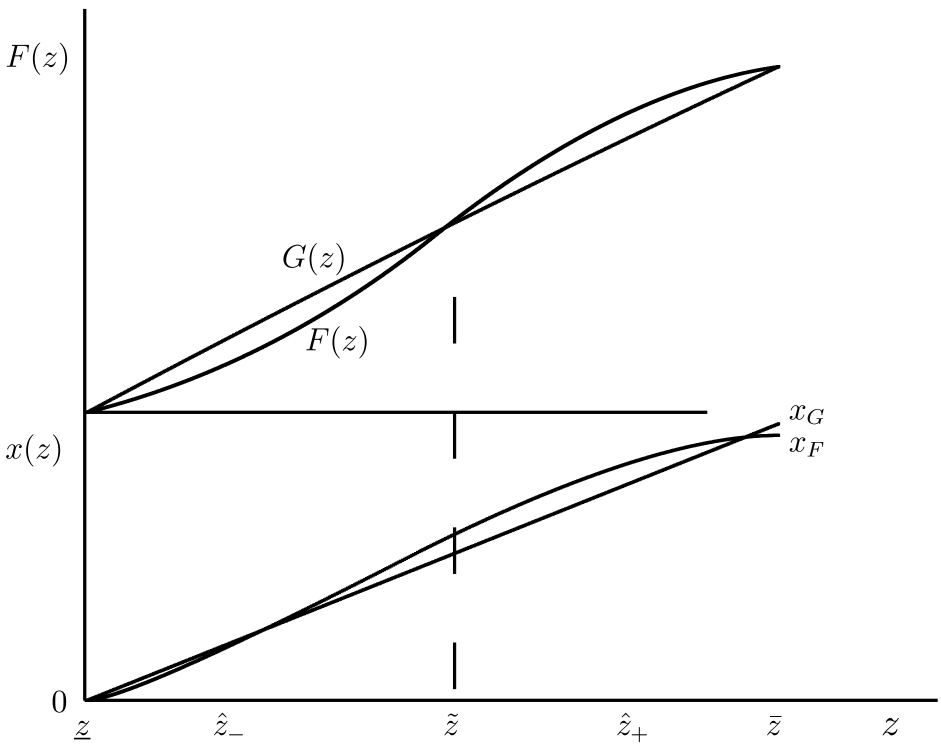

Figure 2: Comparative Statics for a Weakly Competitive Game.

Wolinsky, each firm has private information about its marginal cost c, which is an independent draw from the distribution $F(c)$ which has support $[\underline{c}, \bar{c}]$. Expected profits for a firm with costs c from charging a price $p(\hat{c})$ when the other sellers use the symmetric strategy $p(c)$ are thus

$$V(p(\hat{c}), c, c_{-i}) = \frac{N}{n}(p(\hat{c}) - c)\left(q + n(1-q)(1-F(\hat{c}))^{n-1}\right).$$

This falls into Class 2 and gives rise to the differential equation

$$p'(c) = (p - c)\left(\frac{n(n-1)(1-q)f(c)(1-F(c))^{n-2}}{n(1-q)(1-F(c))^{n-1} + q}\right), \quad p(\bar{c}) = \bar{p}. \tag{13}$$

Define $R(c) = (1 - F(c))^{n-1}$ and use $r(c) = R'(c)$. Then, we have $\psi(p, c) = p - c$ and $m(c) = -n(1-q)r(c)/(n(1-q)R(c) + q)$. Further define

$$P(c) = \frac{n(1-q)R_F(c) + q}{n(1-q)R_G(c) + q} \tag{14}$$

Proposition 6 *Suppose $p_F(c)$ and $p_G(c)$ are the equilibrium pricing functions arising from the differential equation (12) for distributions $F(z)$ and $G(z)$, respectively. If $F(z) \succ_{MLR} G(z)$, then $p_F(c) < p_G(c)$ on $(\hat{c}, \bar{c})$ where $\hat{c}$ is the crossing point of $m_F(z)$ and $m_G(z)$ (maximum of $P(c)$).*

Proof: It can be established by similar inequality to that in (17), that if $F(z) \succ_{MLR} G(z)$ then the ratio $r_F(c)/r_G(c)$ is increasing on $(\underline{c}, \bar{c})$. Note that $P(\underline{c}) = P(\bar{c}) = 1$ and that as F stochastically dominates G, we have $F(c) \leq G(c)$ and $P(c) \geq 1$ for $c \in (\underline{c}, \bar{c})$. It is easily checked that at any point where $P'(c) = 0$, then $P(c) = r_F(c)/r_G(c)$. And since $r_F(c)/r_G(c)$ is increasing there can be only one turning point for $P(c)$, that is, it is unimodal with a unique maximum at $\hat{c}$. Note that $m_F(c) > m_G(c)$ if and only if $P'(c) < 0$. So, for $c > \hat{c}$, we have $m_F(c) > m_G(c)$ and the result follows from Theorem 2. ∎

We would expect a decrease in costs to make the market uniformly more competitive. However, here a stochastically lower distribution of costs induces the high cost firms (firms with costs greater than $\hat{c}$) to charge higher prices. The reason for this is the presence of the uninformed consumers, who assure any firm a minimum demand of qN/n, and make this game only "weakly competitive". With a lower distribution of costs, a firm at any given level of costs will be less likely to win the competition to name the lowest price and attract the informed consumers. If one's costs are high, the chances of winning can be so low, that it may be better to give up the chase. Compare this with the Bertrand model considered in the previous section (or equivalently this model with $q = 0$). There, charging a high price ensures only zero profits.

Note, however, there is another change that produces a reduction in prices for all firms. Since $\partial m(c)/\partial q = r(c)/(n(1-q)R(c)+q)^2 < 0$ for $c \in (\underline{c}, \bar{c})$, a fall in q, the proportion uninformed, leads to a rise in $m(c)$ and lower prices almost everywhere by a simple application of Theorem 2.

7 Conclusions

In this paper, we investigate two new types of comparative statics in games of incomplete information, both of which give rise to non-monotone results. First, we show that refinements of second order stochastic dominance are suitable for comparative statics in games of incomplete information, but are not in general sufficient for monotonicity. Second, we identify a class of games, including all-pay auctions and an incomplete information version of Varian's (1980) model of price dispersion, where even a stochastically higher distribution of types does not necessarily lead to uniformly more aggressive play.

In this paper, we surveyed some stochastic orderings used to rank distributions in terms of dispersion. We also applied them to comparative statics analysis. We hope that they will find further similar applications. First, there has been a recent interest in the effect of changes in inequality in the degree of social competition (Samuelson (2004); Hopkins and Kornienko (2004)). Second, we do not investigate asymmetric auctions or other games in this paper. However, the orderings in terms of dispersion used here should also be useful, for example, in determining the effects of one player having more precise information than other bidders.

Appendix

Proof of Proposition 1: First, we can write an agent's expected utility conditional on bidding x as $\Pi U(z-x)+(1-\Pi)\,U(-Ix)$ where Π is the probability of winning. It can then be verified that by the assumption that $U'' \leq 0$, the agent's indifference curves in (x,Π) space have the single crossing property of Maskin and Riley (2003), whether $I=1$ or $I=0$. Their Lemma 2 (also Proposition 1 of Maskin and Riley (2000a)), which establishes that any best response in a first price auction with independent types is monotone in one's type, also can be applied in the all-pay case. Specifically, the proof follows from the finding that the derivative of expected utility with respect to one's type $\partial V/\partial z$ is increasing in one's bid x. Since in the games we consider here $\partial V/\partial z$ is the same whether $I=1$ or $I=0$, and the case of $I=0$ is covered by their original result, the result also holds for $I=1$.

Suppose that the symmetric equilibrium strategy $x(z)$ is not strictly increasing so that $\breve{x}=x(z_0)=x(z_1)$ for some $z_0<z_1$. Then, given that if all others adopt the symmetric strategy $x(z)$, $\Pi(\hat{x})=F^{n-1}(\hat{z})$ is the probability of winning with a bid $\hat{x}=x(\hat{z})$, we have $\Pi(\breve{x})>\lim_{x\uparrow\breve{x}}\Pi(x)$, that is, there is a discrete jump in the probability of winning at $\breve{x}$. But U is continuous in x, so there must exist an increase in x sufficiently small that the consequent increase in the probability of winning will be greater than any decrease in utility U. That is, there is a profitable deviation, which must be feasible for an agent with type z_1 as $z_1>z_0\geq\breve{x}$. Hence, a symmetric equilibrium strategy must be strictly increasing.

Furthermore, in a symmetric equilibrium the equilibrium strategy $x(z)$ will be continuous. Suppose not, so there is a jump upwards in the equilibrium strategy at some $\breve{z}$ so that $\lim_{z\to\breve{z}}x(z)=\hat{x}\neq x(\breve{z})$. Note, that as $x(z)$ is strictly increasing, despite the discontinuity at $\breve{z}$ we have $\lim_{x\to x(\breve{z})}\Pi(x)=\Pi(x(\breve{z}))=\Pi(\hat{x})$. An individual of type $\breve{z}+\epsilon$ for $\epsilon>0$ who decreases her bid from $x(\breve{z}+\epsilon)$ to $\min[\hat{x},x(\breve{z})]$, that is to the level at the bottom of the jump, will gain a discrete increase in utility U. But by the continuity of Π on $[x(\breve{z}),x(\breve{z}+\epsilon)]$, there must exist an ϵ sufficiently small that the consequent decrease in Π will be smaller than the increase in direct utility U. That is, there is a profitable deviation. It is also possible to establish differentiability of $x(z)$ using standard but lengthy arguments (see, for example, Maskin and Riley (1984)). Then, for any $z\in(\underline{z},\bar{z}]$ the first order conditions for a maximum give rise to the differential equation (2). Given the differentiability of $x(z)$, $F(z)$ and $U(z)$, the right-hand side of (2) is continuously differentiable. Existence and uniqueness of a solution then follows from the fundamental theorem of differential equations. ∎

Proof of Theorem 1: Let us write $x_F(z)-x_G(z)$ as follows:

$$\begin{aligned} x_F(z)-x_G(z) &= \int_{\underline{z}}^{z}[\psi(x_F(t),t)m_F(t)-\psi(x_G(t),t)m_G(t)]dt = \\ &= \int_{\underline{z}}^{z}[\psi(x_F(t),t)m_F(t)-\psi(x_F(t),t)m_G(t)+\psi(x_F(t),t)m_G(t)-\psi(x_G(t),t)m_G(t)]dt = \end{aligned}$$

$$= \int_{\underline{z}}^{z} \psi(x_F(t),t)[m_F(t) - m_G(t)]dt + \int_{\underline{z}}^{z} [\psi(x_F(t),t) - \psi(x_G(t),t)]m_G(t)dt$$

and further rewrite the above as:

$$x_F(z) - x_G(z) + \int_{\underline{z}}^{z} [\psi(x_G(t),t) - \psi(x_F(t),t)]m_G(t)dt = \int_{\underline{z}}^{z} \psi(x_F(t),t)[m_F(t) - m_G(t)]dt \tag{15}$$

Let us first show that if $x_F(z)$ and $x_G(z)$ do cross on $(\underline{z}, \underline{z}+\delta)$, then $x_F(z)$ crosses $x_G(z)$ from below. This easy to see from the equation (4), as it implies that at any point $z_\times$ such that $x_F(z_\times) = x_G(z_\times)$, we have that

$$\frac{x'_F(z_\times)}{x'_G(z_\times)} = \frac{m_F(z_\times)}{m_G(z_\times)} \tag{16}$$

That is, as $m_F(z_\times) > m_G(z_\times)$ for any $z_\times \in (\underline{z}, \underline{z}+\delta)$ we have $x'_F(\times) > x'_G(\times)$. This implies that there is at most a single crossing of $x_F(z)$ and $x_G(z)$ on $(\underline{z}, \underline{z}+\delta)$.

Consequently, there are three possible cases. First, $x_F(z) > x_G(z)$ on $(\underline{z}, \underline{z}+\delta)$. Second, $x_F(z) < x_G(z)$ on $(\underline{z}, \underline{z}+\delta)$. Third, $x_F(z) < x_G(z)$ on $(\underline{z}, z_1)$ for $z_1 < \underline{z}+\delta$ where z_1 is the unique crossing point of the two solutions. Note that the second and third possibilities both imply that $x_F(z) < x_G(z)$ on $(\underline{z}, \underline{z}+\epsilon)$ for some $\epsilon \in (0, \delta]$. Given that $m_F(z) > m_G(z)$ on $(\underline{z}, \underline{z}+\delta)$, the RHS of (14) is positive on the interval. In this case, as $x_F(z) < x_G(z)$ and ψ is decreasing in x, $\psi(x_F(z), z) > \psi(x_G(z), z)$ on $(\underline{z}, \underline{z}+\epsilon)$ so that the LHS of (14) is negative for all $z < \underline{z}+\epsilon$, which is a contradiction. Thus, if $m_F(z) > m_G(z)$ on $(\underline{z}, \underline{z}+\delta)$, only the first case is possible, that is, $x_F(z) > x_G(z)$ on $(\underline{z}, \underline{z}+\delta)$.

Lastly, examining equation (15), if $x_F(z)$ and $x_G(z)$ do cross, then at every point of crossing $z_\times$, $x_F(z)$ crosses $x_G(z)$ from above (so that $x'_F(z_\times) < x'_G(z_\times)$) only if $m_F(z_\times) < m_G(z_\times)$ and $x_F(z)$ crosses $x_G(z)$ from below (so that $x'_F(z_\times) > x'_G(z_\times)$) only if $m_F(z_\times) > m_G(z_\times)$. This implies that any point of sign change in $x_F(z) - x_G(z)$ is always to the right of some point of sign change in $m_F(z) - m_G(z)$, so that $S(x_F(z) - x_G(z)) \leq S(m_F(z) - m_G(z))$. ∎

Proof of Theorem 3: (i) If $I = 0$, and we take $U(0) = 0$ then the differential equation reduces to (16). Now, by Corollary 1, we have $f(z)/F(z) > g(z)/G(z)$ almost everywhere on $(\underline{z}, \hat{z}_P)$ and the result follows from application of Theorem 1.

(ii) If $F \succ_{ULR} G$ then $F(z)/G(z)$ is unimodal from Corollary 1, and $F(z) < G(z)$ on $(\underline{z}, \tilde{z})$ with $\tilde{z} \leq \bar{z}$. Clearly also $F^{n-1}(z) < G^{n-1}(z)$ on $(\underline{z}, \tilde{z})$. Now, $h_F(z)$ is the density function for the distribution $F^{n-1}(z)$ as $h_G(z)$ is for $G^{n-1}(z)$. Since $h_F(z)$ and $h_G(z)$ are both density functions and $F^{n-1}(\tilde{z}) = G^{n-1}(\tilde{z})$, it cannot be that one density is always bigger than the other on $(\underline{z}, \tilde{z})$. The ratio $h_F(z)/h_G(z) = f(z)F^{n-2}(z)/(g(z)G^{n-2}(z))$ is increasing if

$$\frac{f'(z)}{f(z)}\frac{f(z)}{F(z)} > \frac{g'(z)}{g(z)}\frac{g(z)}{G(z)} \tag{17}$$

It is then easy to verify from Corollary 1 that if $F \succ_{ULR} G$, then $h_F(z)/h_G(z)$ is increasing on $(\underline{z}, \hat{z}_L)$. This implies that $h_F(z)$ crosses $h_G(z)$ exactly once and from below. Thus, for some $\epsilon > 0$, $h_F(z) < h_G(z)$ on $(\underline{z}, \underline{z} + \epsilon)$. Now we would like to apply Theorem 1, but we have the complication that here the function $\psi = (U(z-x) - U(-x))/(U'(z-x)F^{n-1}(z) + U'(-x)(1-F^{n-1}(z)))$ depends on the distribution function $F(z)$ as well as z and x, or here we have $\psi(x(z), z, F(z))$. However, if ψ is increasing in $F(z)$, as it is here, it can be shown that the argument proceeds as before. First, given that $F(z) < G(z)$ on $(\underline{z}, \tilde{z})$ at any point of crossing of $x_F(z)$ and $x_G(z)$ in that interval, x_G must cross from below, and there can be only one such crossing. Thus, there are two possibilities, either the result is proved, or there is an $\epsilon > 0$ such that $x_F(z) > x_G(z)$ on $(\underline{z}, \underline{z} + \epsilon)$. But this would imply that $\psi(x_F(z), z, F(z)) < \psi(x_G(z), z, G(z))$ on that interval, which together with $h_F(z) < h_G(z)$, would imply that $x'_F(z) < x'_G(z)$ on $(\underline{z}, \underline{z} + \epsilon)$, which is a contradiction. ∎

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