Diffeomorphism equivalence and permutation equivalence

Jeffrey Ketland

May 12, 2009

Suppose that $\mathcal{M} = (A, R_1, ..., R_n)$ is a structure. That is, A is a non-empty set, and, for each $i = 1$ to n, $R_i \subseteq A^k$, for some k. Suppose that $\pi : A \to A$ is a permutation of A (i.e., a bijection of A to itself). If $R_i \subseteq A^k$ is a k-place relation on A, then let $\pi(R_i)$ be $\{(\pi(a_1), ..., \pi(a_k)) : (a_1, ..., a_n) \in R_i\}$. Then,

Definition 1 $\mathcal{M}^\pi =_{df} (A, \pi(R_1), ..., \pi(R_n))$. *The structure* $\mathcal{M}^\pi$ *is called the* ***Quine transform*** *of* $\mathcal{M}$.

One can see that $\pi : \mathcal{M} \to \mathcal{M}^\pi$ is an *isomorphism*. So, a structure $\mathcal{M}$ and its Quine transform $\mathcal{M}^\pi$ are always isomorphic. This is, of course, a very cheap way of obtaining isomorphic structures! One simply permutes the base set any way one likes, and then "drags" all other relations along with the permutation of points.

The reason for the term "Quine transform" is that the idea of "*permuting*" the elements of a structure in this way has a certain resonance with W.V. Quine's notion of a "proxy function" from his work on "inscrutability of reference" and "ontological relativity". This is a kind of semantic thesis. We consider an interpreted language $\mathbb{L} = (\mathcal{L}, \mathcal{M})$ and any bijection $\pi : A \to A$, where A is the base set of $\mathcal{M}$. Then we obtain another interpreted language $\mathbb{L}^\pi = (\mathcal{L}, \mathcal{M}^\pi)$.

Quine's philosophical claim is:

> **Quine Equivalence** (of interpreted languages)
> The interpreted languages $\mathbb{L}$ and $\mathbb{L}^\pi$ are, in some deep sense, *indistinguishable*.

As Quine sometimes explains it, if s is a speaker, then there can be no "fact of the matter" whether s "speaks" $\mathbb{L}$ or s speaks $\mathbb{L}^\pi$. Both languages may contain a closed $\mathcal{L}$-term t. But in $\mathbb{L}$, t refers to $t^{\mathcal{M}}$, while in $\mathbb{L}^{\pi}$, t refers to $t^{\mathcal{M}^{\pi}}$ (which is, of course, $\pi(t^{\mathcal{M}})$). Quine insists that there is no "fact of the matter" here whether the speaker is referring to $t^{\mathcal{M}}$ or to $\pi(t^{\mathcal{M}})$. There is an air of mystery about this, of course. After all, the reference functions built in to $\mathbb{L}$ and $\mathbb{L}^{\pi}$ are mathematically different, although they are obtained by simply permuting the domain. In any case, just as we might call the interpreted languages $\mathbb{L}$ and $\mathbb{L}^{\pi}$ "Quine-equivalent", we might also call the structures $\mathcal{M}$ and $\mathcal{M}^{\pi}$ "Quine-equivalent". So, a structure $\mathcal{M}$ and its Quine-transform $\mathcal{M}^{\pi}$, under the bijection $\pi : A \to A$, are Quine-equivalent.

It is sometimes claimed that general relativity involves a certain kind of "gauge symmetry". The idea is this. Suppose that $\mathcal{M} = (M, g_{ab}, \phi_1, ..., \phi_n)$ is a spacetime structure, where M is a differentiable manifold, g_{ab} is a distinguished (0,2) tensor on M (that is, the metric tensor), and the ϕ_i are "matter fields" on M.[1] Suppose that $h : M \to M$ is a diffeomorphism of M to itself. That is, h is an automorphism of M. Let $\mathcal{M}^h$ be the result of applying this diffeomorphism also to the metric tensor g_{ab} and to all the matter fields. I.e., $\mathcal{M}^h = (M, (g_{ab})^h, (\phi_1)^h, ..., (\phi_n)^h)$, where, for any point $p \in M$, $(g_{ab})^h(p) =_{df} g_{ab}(h(p))$, and similarly for the matter fields ϕ_i. The equivalence claim is sometimes called "Leibniz equivalence":

> **Leibniz Equivanence** (of spacetimes)
> Let M be the *base manifold* of the spacetime structure $\mathcal{M}$. Let $h : M \to M$ be a diffeomorphism. Then $\mathcal{M}$ and $\mathcal{M}^h$ "*represent the same physical situation*".

Note that it is nearly trivial that if $\mathcal{M}$ is a solution of Einstein's field equations, then its Leibniz transform $\mathcal{M}^h$ is *a* solution too. The point of Leibniz equivalence is that $\mathcal{M}$ and $\mathcal{M}^h$ are, physically, the *same* solution.

However, it seems to me that Leibniz equivalence might really be a consequence of a stronger "gauge symmetry" which permits *complete arbitrariness in the permutation* of the base set. That is, one can permute the points of the base set anyway one likes; and, so long as one also transforms *all* the structure accordingly (including the topological structure), then the resulting structure is again physically equivalent to the first. Diffeomorphisms are a special kind of permutation of the base set. And diffeomorphism equivalence is then really a special case of a quite general permutation equivalence!

To make this more explicit, we must exhibit more fully the structure of a differentiable manifold, M. Clearly M must have a base set, which we

[1]Usually we demand that these tensor fields satisfy certain conditions – Einstein's equations. That is, the Ricci tensor R_{ab} constructed from the metric g_{ab} is related in a certain way to the energy-momentum tensor T_{ab} constructed from the matter fields. For our purposes, the dynamical equations are entirely irrelevant.

call call A. Then M can be described as a structure of the form $(A, \mathcal{C})$, where $\mathcal{C}$ is an atlas of charts. Or M can be described as a topological space $(A, \mathcal{T})$, where $\mathcal{T}$ is a collection of subsets of A satisfying certain conditions. I shall choose the latter. So, a differentiable manifold M is a topological space $(A, \mathcal{T})$ satisfying certain conditions. Now suppose that $M' = (A', \mathcal{T}')$ is another differentiable manifold. What does it mean to say that M and M' are isomorphic? Well, it means that there is a bijection $f : A \to A'$ such that for any $O \subseteq A$, $O \in \mathcal{T}$ iff $f(O) \in \mathcal{T}'$, where $f(O) =_{df} \{f(p) : p \in O\}$.[2]

Now, given a manifold $M = (A, \mathcal{T})$, and any permutation $\pi : A \to A$, then we define:

Definition 2 $\mathcal{T}^{\pi} =_{df} \{f(O) : O \in \mathcal{T}\}$.

Definition 3 $M^{\pi} =_{df} (A, \mathcal{T}^{\pi})$.

By construction, M and M^{π} are isomorphic (under the permutation $\pi : A \to A$). Indeed, π is a diffeomorphism from M to M^{π}. However, and note well, such a permutation $\pi : A \to A$ need *not* be a diffeomorphism from M to *itself*. That is, π need not be an *automorphism* of M. Indeed, usually it isn't. However, even though π may not be an automorphism of M, π *is* a diffeomorphism from M to its Quine transform, M^{π}. Furthermore,

Definition 4 *Let* $\mathcal{M} = (M, g_{ab}, \phi_1, ..., \phi_n)$ *be a spacetime structure. Then* $\mathcal{M}^{\pi} =_{df} (M^{\pi}, (g_{ab})^{\pi}, (\phi_1)^{\pi}, ..., (\phi_n)^{\pi})$.

Then $\mathcal{M}^{\pi}$, the Quine transform of $\mathcal{M}$, is indeed isomorphic to $\mathcal{M}$.

There seems no reason to think that $\mathcal{M}$ and its Quine transform, $\mathcal{M}^{\pi}$, represent different physical situations, even though the points in $\mathcal{M}$ have been *arbitrarily permuted* by π! So, perhaps we are led to a stronger gauge equivalence principle,

> **Quine Equivalence** (of spacetimes)
> Let A be the base set of spacetime structure $\mathcal{M}$ and let $\pi : A \to A$ be a permutation. Then $\mathcal{M}$ and $\mathcal{M}^{\pi}$ "*represent the same physical situation*".

[2]This in fact, is equivalent to saying that $f : M \to M'$ is a *homeomorphism*. The definitions looks different from the usual one, but as long as f is a bijection, one can show that the two directions of biconditional express that both f and f^{-1} are continuous. Diffeomorphisms are homeomorphisms between differentiable manifolds. If topological spaces are regarded as 2-sorted first-order structures, of the form $((A, \mathcal{T}), \in)$, then a homeomorphism is a 2-sorted isomorphism.

Quine Equivalence implies Leibniz Equivalence. But the former is stronger. The difference between the two is that the "Leibniz transform" of a spacetime $\mathcal{M}$ requires that the permutation π be an *automorphism* of the base manifold M, while the "Quine transform" of $\mathcal{M}$ permits *any old permutation* π of the *base set*. Of course, permutations of a manifold are not usually going to be diffeomorphisms! They will hugely wreck continuity relations amongst points. However, the significance of diffeomorphisms arises from demanding that the permutation $\pi : M \to M$ of the points of M be an *automorphism*. But it's difficult to see, in that case, why we should not demand that permutations involved for a spacetime structure $\mathcal{M} = (M, g_{ab}, \phi_1, ..., \phi_n)$ should also be automorphisms of $\mathcal{M}$. And of course, usually if $h : M \to M$ is a diffeomorphism of the base manifold, then h is *not* also an automorphism of the whole structure $\mathcal{M}$, including the metric and the matter fields. But we do not do this in usual discussion of Leibniz symmetry in general relativity. Rather, we demand that we have an automorphism of the base manifold M – i.e., a diffeomorphism – but then everything else is "dragged along" by the diffeomorphism. In other words, we distinguish between topological structure on the base set, and other structure over this. I'm not sure if this can be motivated. Perhaps it can, and I am missing something – perhaps something obvious.

But if it can't, then there seems no reason not to permute *everything*, and then to "drag" everything along with the permutation. And if this is correct, then the underlying gauge symmetry of general relativity is a kind of *Quinian permutation equivalence*. In some sense, which is terribly difficult to articulate, the identities of points of the base set have no significance at all. Only the "structural patterns" imposed on them have significance.